Simple proofs of classical results on zeros of $J_\nu(x)$ and $J'_\nu(x)$

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Abstract

The Bessel functions $J_\nu(x)$ and their derivatives $J'_\nu(x)$ can be represented by infinite series and infinite products. Using these representations we give very simple proofs for known results concerning the zeros of the above functions.

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1 Introduction

It is well known [4, 5] that the Bessel function $J_\nu(x)$ and its derivative $J'_\nu(x)$ can be represented by the infinite series:

\[ J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-\frac{x^2}{4})^n}{n!\Gamma(\nu + n + 1)}, \quad \nu > -1 \]  

(1.1)

\[ J'_\nu(x) = \frac{1}{2} \left(\frac{x}{2}\right)^{\nu-1} \sum_{n=0}^{\infty} \frac{(-\frac{x^2}{4})^n (2n + \nu)}{n!\Gamma(\nu + n + 1)}, \quad \nu > 0 \]  

(1.2)

as well as by infinite products:

\[ J_\nu(x) = \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu + 1)} \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{j_{\nu,n}^2}\right), \quad \nu > -1 \]  

(1.3)

and

\[ J'_\nu(x) = \frac{1}{2} \left(\frac{x}{2}\right)^{\nu-1} \frac{1}{\Gamma(\nu)} \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(j'_{\nu,n})^2}\right), \quad \nu > 0 \]  

(1.4)

respectively. By $j_{\nu,n}$ and $j'_{\nu,n}$, $n = 1, 2, ...$ we indicate the n-th positive zeros of $J_\nu(x)$ and $J'_\nu(x)$ respectively. Using only these representations for $J_\nu(x)$ and $J'_\nu(x)$ we obtain very easily well known [1, 2, 3, 5] results concerning the zeros of these functions.

2 Results on the zeros of $J_\nu(x)$

By equating the right hand side of (1.1) and (1.3) we obtain

\[ \sum_{n=0}^{\infty} \frac{(-\frac{x^2}{4})^n}{n!\Gamma(\nu + n + 1)} = \frac{1}{\Gamma(\nu + 1)} \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{j_{\nu,n}^2}\right). \]  

(2.1)
Let us consider the first terms of the series on the left and the first terms of the products on the right, so:

\[
\frac{1}{\Gamma(\nu + 1)} - \frac{1}{4} \frac{x^2}{\Gamma(\nu + 2)} + \frac{1}{4^2} \frac{x^4}{2! \Gamma(\nu + 3)} - \frac{1}{4^3} \frac{x^6}{3! \Gamma(\nu + 4)} + \ldots
\]  

(2.2)

\[
= \frac{1}{\Gamma(\nu + 1)} (1 - \frac{x^2}{J_{\nu,1}^2}) (1 - \frac{x^2}{J_{\nu,2}^2}) (1 - \frac{x^2}{J_{\nu,3}^2}) \ldots
\]  

(2.3)

Using the equality \( \Gamma(x + 1) = x \Gamma(x) \), it becomes:

\[
1 - \frac{1}{4} \frac{x^2}{\nu + 1} + \frac{1}{4^2} \frac{x^4}{2!(\nu + 1)(\nu + 2)} - \frac{1}{4^3} \frac{x^6}{3!(\nu + 1)(\nu + 2)(\nu + 3)} + \ldots
\]  

(2.4)

\[
= (1 - \frac{x^2}{J_{\nu,1}^2})(1 - \frac{x^2}{J_{\nu,2}^2})(1 - \frac{x^2}{J_{\nu,3}^2}) \ldots
\]  

(2.5)

1) By equating the coefficients of \( x^0, x^2, x^4, \ldots \) of (2.5) we obtain respectively

\[
1 = 1,
\]  

(2.6)

\[
\frac{1}{4(\nu + 1)} = \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2},
\]  

(2.7)

\[
\frac{1}{4^2 2!(\nu + 1)(\nu + 2)} = \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} \sum_{k=1, k \neq n}^{\infty} \frac{1}{j_{\nu,k}^2}
\]  

(2.8)

Taking in account (2.7) the sums of the right hand side of (2.8) can be written

\[
\sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} \sum_{k=1, k \neq n}^{\infty} \frac{1}{j_{\nu,k}^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} \left( \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^2} - \frac{1}{j_{\nu,n}^2} \right)
\]  

(2.9)

\[
= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} \left( \frac{1}{4(\nu + 1)} - \frac{1}{j_{\nu,n}^2} \right) = \frac{1}{2} \left[ (\frac{1}{4(\nu + 1)})^2 - \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^4} \right]
\]  

(2.10)

so, the equation (2.8) takes the form

\[
\frac{1}{4^2 2!(\nu + 1)(\nu + 2)} = \frac{1}{2} \left[ (\frac{1}{4(\nu + 1)})^2 - \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^4} \right]
\]  

(2.11)

or

\[
\sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^4} = \frac{1}{2^4(\nu + 1)^2(\nu + 2)}
\]  

(2.12)

**Remark 2.1.** If we continue using the analogous procedure by equating the coefficients of \( x^6, \ldots \), we’ll obtain the sums \( \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^k}, \ k = 3, \ldots \)
Remark 2.2. We mention that the sums $\sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}}$, $k = 1, 2, 3, \ldots$ are well known [1, 2, 3, 5] but their proof is much more complicated.

Remark 2.3. It is obvious that using the sums $\sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}}$, $k = 1, 2, 3, \ldots$ we obtain [5] known inequalities for the first zero of $J_{\nu}(x)$. For example using (2.12) we obtain the lower bound $j_{\nu,1}^2 > 4(\nu + 1)(\nu + 2)^{1/2}$, for $\nu > -1$.

2) Putting $\nu = 1/2$ in (2.5) and since $j_{1/2,n} = n\pi$, it becomes:

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2^2\pi^2})(1 - \frac{x^2}{3^2\pi^2})\ldots$$

(2.13)

or

$$\sin x = x \Pi_{n=1}^{\infty} (1 - \frac{x^2}{n^2\pi^2})$$

(2.14)

which is the known [4] infinite product expansion for $\sin x$.

3) Similarly, by putting $\nu = -1/2$ in (2.5) and since $j_{-1/2,n} = (2n - 1)\pi/2$, it becomes:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = (1 - \frac{4x^2}{\pi^2})(1 - \frac{4x^2}{3^2\pi^2})(1 - \frac{4x^2}{5^2\pi^2})\ldots$$

(2.15)

or

$$\cos x = \Pi_{n=1}^{\infty} (1 - \frac{4x^2}{(2n - 1)^2\pi^2})$$

(2.16)

which is the known [4] infinite product expansion for $\cos x$.

4) We put $iy$ instead of $x$ in (2.5), so it becomes:

$$1 + \frac{1}{4}y^2 \frac{1}{\nu + 1} + \frac{1}{4^2}y^4 \frac{1}{2!(\nu + 1)(\nu + 2)} + \frac{1}{4^3}y^6 \frac{1}{3!(\nu + 1)(\nu + 2)(\nu + 3)} + \ldots$$

(2.17)

$$= (1 + \frac{y^2}{J_{\nu,1}})(1 + \frac{y^2}{J_{\nu,2}})(1 + \frac{y^2}{J_{\nu,3}})\ldots$$

(2.18)

and $y$ are the zeros of the modified Bessel function $I_{\nu}(y)$. By putting $\nu = 1/2$ in (2.18) we have

$$1 + \frac{y^2}{3!} + \frac{y^4}{5!} + \frac{y^6}{7!} + \ldots = (1 + \frac{y^2}{\pi^2})(1 + \frac{y^2}{2^2\pi^2})(1 + \frac{y^2}{3^2\pi^2})\ldots$$

(2.19)

or

$$\sin hy = y \Pi_{n=1}^{\infty} (1 + \frac{y^2}{n^2\pi^2})$$

(2.20)

which is the known [4] infinite product expansion for $\sin hy$.

5) Similarly, by putting $\nu = -1/2$ in (2.18) we have:

$$1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \frac{y^6}{6!} + \ldots = (1 + \frac{4y^2}{\pi^2})(1 + \frac{4y^2}{3^2\pi^2})(1 + \frac{4y^2}{5^2\pi^2})\ldots$$

(2.21)

or

$$\cos hy = \Pi_{n=1}^{\infty} (1 + \frac{4y^2}{(2n - 1)^2\pi^2})$$

(2.22)

which is the known [4] infinite product expansion for $\cos hy$. 
Remark 2.4. From (2.14) we also obtain the well known [4] result that \( \lim_{x \to 0} \frac{\sin x}{x} = 1. \)

Remark 2.5. The equations (2.7) and (2.12) for \( \nu = 1/2 \) and \( \nu = -1/2 \) give the known summable series
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}
\]
respectively.

3 Results on the zeros of \( J'_\nu(x) \)

By equating the right hand side of (1.2) and (1.4) we obtain
\[
\sum_{n=0}^{\infty} \frac{(-x^2)^n(2n+\nu)}{n!\Gamma(\nu+n+1)} = \frac{1}{\Gamma(\nu)} \Pi_{n=1}^{\infty} \left(1 - \frac{x^2}{(j'_\nu,n)^2}\right).
\]  

We are working similarly as in section 2, so, we consider the first terms of the series on the left and the first terms of the products on the right, so:
\[
\frac{\nu}{\Gamma(\nu+1)} \frac{x^2}{4} \frac{(2+\nu)}{\Gamma(\nu+2)} + \frac{x^4}{4^2 2!\Gamma(\nu+3)} \frac{(4+\nu)}{\Gamma(\nu+1)} - \frac{x^6}{4^3 3!\Gamma(\nu+4)} \frac{(6+\nu)}{\Gamma(\nu+1)} + \ldots
\]  

and using the equality \( \Gamma(x+1) = x\Gamma(x) \), it becomes:
\[
1 - \frac{x^2}{4} \frac{(2+\nu)}{\nu(\nu+1)} + \frac{x^4}{4^2} \frac{(4+\nu)}{2\nu(\nu+1)(\nu+2)} - \frac{x^6}{4^3} \frac{(6+\nu)}{3\nu(\nu+1)(\nu+2)(\nu+3)} + \ldots
\]  

By equating the coefficients of \( x^0, x^2, x^4, \ldots \) we obtain respectively
\[
1 = 1,
\]  
\[
\frac{(2+\nu)}{4\nu(\nu+1)} = \sum_{n=1}^{\infty} \frac{1}{(j'_\nu,n)^2},
\]  
\[
\frac{(4+\nu)}{4^2 2!\nu(\nu+1)(\nu+2)} = \sum_{n=1}^{\infty} \frac{1}{(j'_\nu,n)^2} \sum_{k=1,k\neq n}^{\infty} \frac{1}{(j'_\nu,k)^2}
\]  

As in the previous section, the sum in right hand side of (3.8) can be written
\[
\sum_{n=1}^{\infty} \frac{1}{(j'_\nu,n)^2} \sum_{k=1,k\neq n}^{\infty} \frac{1}{(j'_\nu,k)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(j'_\nu,n)^2} \left( \sum_{k=1}^{\infty} \frac{1}{(j'_\nu,k)^2} - \frac{1}{(j'_\nu,n)^2} \right)
\]  

so we obtain
\[
\sum_{n=1}^{\infty} \frac{1}{(j'_\nu,n)^4} = \frac{(\nu^2 + 8\nu + 8)}{4^2 \nu^2 (\nu+1)(\nu+2)}.
\]
Remark 3.1. If we continue using the analogous procedure by equating the coefficients of $x^6$, ..., we’ll obtain the sums $\sum_{n=1}^{\infty} \frac{1}{(j_{\nu,n})^{2k}}$, $k = 3$, ....

Remark 3.2. We mention that the sums $\sum_{n=1}^{\infty} \frac{1}{(j_{\nu,n})^{2k}}$, $k = 1, 2, 3, ...$ are well known [1, 3] but their proof is much more complicated.

References


